Enumeration of branched coverings of the sphere

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1 Introduction

My goal is to count the number of connected ramified coverings of the sphere, with given ramification data. This can be interpreted in terms of tilngs of compact surfaces by triangles, squares and hexagons. The computation of the generating series of these numbers requires the language of Hurwitz theory, and, more specifically, of representation theory of the symmetric group. The important point is that these series are quasimodular for $\Gamma_1(N)$. As generators of the algebra of quasimodular forms are known, I will be able to explicit these series in a known basis. In this internship report, I have two major goals : Firstly, explain the basics of (quasi)modular forms, and, secondly, compute these series. I used Sage throughout my internship to compute these generating functions.

2 Modular Forms

2.1 Motivations

Modular Forms are interesting objects, which are the subject of study in and of themselves. Nevertheless, here I'd like to give some motivations, coming from the field of Riemann surfaces. A *complex tori* is a quotient $\mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$. Two such tori T_1 and $T_2$are isomorphic as Riemann surfaces if and only if there exists some matrix $A \in SL_2(\mathbb{Z})$ such that $\Gamma_1 = A\Gamma_2$. Since any lattice is equivalent to the lattice $\mathbb{Z} + \tau\mathbb{Z}$ where $\tau \in \mathbb{H}$, having a function from the space of complex tori, is the same as having a function of a complex variable, satisfying the *modular invariance property*:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$.

However, this definition is often times too restrictive. For example, if we think of the quotient $SL_2(\mathbb{Z})\backslash\mathbb{H}$, and if we add the point at the end of the cusp to compactify the curve, then, no such non-constant holomorphic function f satisfies the desired property. The definition of modular form will ask that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for some positive integer k, and we will add the desired property that f is "holomorphic at ∞".

Another motivation comes from the field of hyperbolic geometry. The upper half plane, $\mathbb{H}$, with the hyperbolic metric $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$ is a riemannian manifold of constant negative curvature. Its isometries consists of the group $PSL_2(\mathbb{R})$, where $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by the following formula :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

If Γ is a discrete subgroup of $PSL_2(\mathbb{R})$, for example $\Gamma = PSL_2(\mathbb{Z})$, then hyperbolic surfaces are surfaces defined by $S = \Gamma\backslash\mathbb{H}$. If we want to have a holomorphic one form on S, then such a one form ω has to satisfy that, for all$\gamma \in \Gamma$, $\gamma_*\omega = \omega$. But, the differential of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{aligned} \mathrm{d}\gamma : T\mathbb{H} = \mathbb{H} \times \mathbb{C} &\to \mathbb{H} \times \mathbb{C} \\ (z, v) &\mapsto \left(\frac{az + b}{cz + d}, \frac{v}{(cz + d)^2}\right). \end{aligned}$$

Now, $\gamma_* \omega = \omega$ rewrites as $f(z)dz = f(\gamma \cdot z)\frac{dz}{(cz+d)^2}$.

So, asking for a holomorphic one form ω to be defined on a hyperbolic surface is the same as asking that, in local coordinates $\omega = f(z)dz$, f is a modular form of weight 2 for Γ.

2.2 Congruence subgroups

From now on until the end, let N be a positive integer.

Definition 2.1. The principal congruence subgroup of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \operatorname{Mod} N \right\}.$$

Remark 2.2. In particular, $\Gamma(1) = SL_2(\mathbb{Z})$.

Proposition 2.3.

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

Proof. This proof can be found in [3]. One first shows that the morphism $SL_2(\mathbb{Z})/\Gamma(N) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective, and hence an isomorphism, then use the isomorphism given by the chinese remainder theorem : $SL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{i=1}^k SL_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$ where $N = \prod_{i=1}^k p_i^{e_i}$, and compute $SL_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$ by induction on e_i. □

Similarly, we also define $\Gamma_0(N)$ and $\Gamma_1(N)$:

Definition 2.4.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \operatorname{Mod} N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \operatorname{Mod} N \right\}$$

Remark 2.5. We have the inclusion $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$.

Definition 2.6. A subgroup Γ of $SL_2(\mathbb{Z})$ is said to be a *Congruence subgroup* if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{N}^*$. The smallest such N is said to be the *level* of Γ.

Remark 2.7. If Γ is a congruence subgroup, then, for all $\gamma \in SL_2(\mathbb{Z})$, $\gamma^{-1}\Gamma\gamma$ is also a congruence subgroup.

2.3 Definitions

The following definitions are motivated by Section 2.1. For $\gamma \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, we write $\gamma(\tau) := \gamma \cdot \tau$ where the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ is given in 2.1.

Definition 2.8. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. Define the *factor of automorphy* :

$$j(\gamma, \tau) = c\tau + d.$$

Also, for a function $f : \mathbb{H} \to \mathbb{C}$ define the *weight k operator* $[\gamma]_k$ by :

$$f[\gamma]_k(\tau) = j(\gamma, \tau)^{-k} f(\gamma\tau).$$

This operator has the following properties, whose proof are just computations, given in [3, Lemma 1.2.2] :

Proposition 2.9. *For all $\gamma, \gamma' \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,*

1. $j(\gamma\gamma', \tau) = j(\gamma, \gamma'(\tau)) j(\gamma', \tau)$
2. $[\gamma\gamma']_k = [\gamma]_k [\gamma']_k$
3. $\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{|j(\gamma,\tau)|^2}$

Asking for a function to be weight k invariant is not enough for it to be a modular form. We also need a stronger condition, namely "holomorphy at the cusps". Let Γ be a congruence subgroup and $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and weight k invariant with repsect to Γ. Because Γ is a congruence subgroup, it contains some $\Gamma(N)$, and so Γ contains $\gamma = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$. Because $j(\gamma, \cdot) = Id$, and because f is weight k invariant and holomorphic, f is $h\mathbb{Z}$ periodic, for some minimal $h \in \mathbb{N}$. Thus, f can be developped as a Fourier Series :

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2in\pi\tau/h}$$

Definition 2.10. Such an f is said to be *holomorphic at* ∞ if $\forall n < 0$, $a_n = 0$. That is,

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2i\pi\tau/h}.$$

Definition 2.11. Let Γ be a congruence subgroup, and k be an integer. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is said to be a *modular form of weight k with respect to* Γ if

1. $\forall \gamma \in \Gamma$, $f[\gamma]_k = f$
2. $\forall \alpha \in SL_2(\mathbb{Z})$, $f[\alpha]_k$ is holomorphic at ∞.

If, in addition, $a_0 = 0$ in the Fourier expansion of $f[\alpha]_k$ for all $\alpha \in SL_2(\mathbb{Z})$, then f is said to be a *cusp form*.

The set of modular forms of weight k with respect to Γ is denoted $\mathcal{M}_k(\Gamma)$, and the set of cusp forms is $S_k(\Gamma)$, and the quotient of modular forms by cusp forms is the *Eisenstein subspace* $\mathcal{E}_k(\Gamma)$.

The second condition, namely, that $f[\gamma]_k$ is always holomorphic at ∞, may seem hard to show. Indeed, there exists a simpler way to show that such a function satisfies this property :

Proposition 2.12. *Suppose that f is a holomorphic, weight k invariant function with respect to Γ. Suppose, also, that in the fourier expansion of f, there exists constants C and r such that for all $n \in \mathbb{N}$,*

$$|a_n| \leq Cn^r$$

Then, f is a modular form with respect to Γ.

Some examples are now definitely in order. In the following examples, I choose $\Gamma = SL_2(\mathbb{Z})$.

Example 2.13. Let $k \geq 3$. Define

$$G_k(\tau) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}$$

and

$$E_k(\tau) = \sum_{\substack{m,n \in \mathbb{Z} \\ \gcd(m,n)=1}} \frac{1}{(m\tau + n)^k}.$$

Factorising every element of the sum in G_k by the gcd of m and n gives the following formula :

$$G_k(\tau) = \zeta(k)E_k(\tau).$$

A quick computation gives that G_k is weight k invariant, and implies that $G_k = 0$ for odd k.

Using the Poisson summation formula, one finds that the Fourier expansion of G_k is given by :

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

where

$$\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$$

and $q = e^{2i\pi\tau}$ (details of this computation are given in [1]). We'll be introducing a third normalisation, whose goal is to normalize to 1 coefficients before q:

$$\mathbb{G}_k(\tau) = \frac{(k-1)!}{(2i\pi)^k} G_k(\tau)$$

Concretely, such functions are given by the following coefficients :

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + \cdots$$
$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 - \cdots$$
$$E_8(\tau) = 1 + 480q + 61920q^2 + 1050240q^3 + \cdots$$

while the other normalisation is

$$\mathbb{G}_4(\tau) = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + \cdots$$
$$\mathbb{G}_6(\tau) = -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots$$
$$\mathbb{G}_8(\tau) = \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots$$

We have only encountered modular forms of a certain weight, but the notion can be extended to modular forms of *mixedweight*, that is, modular forms which are sums of modular forms of different weights. These modular forms are part of the space

$$\mathcal{M}(\Gamma) = \bigoplus_{k \in \mathbb{N}} \mathcal{M}_k(\Gamma).$$

Notice also that, in none of these examples, I used E_2. That is because while the sum defining G_2 converges, it doesn't absolutely converge, making it impossible to show that E_2 is weight 2 invariant. And indeed, it isn't weight 2 invariant, but it is "almost" weight 2 invariant, and is in fact an example of what is going to be called a "quasimodular form", as defined in the next subsection.

2.4 Quasimodular forms

In this subsection, I will define quasimodular forms. These are defined in [6]. Let Γ be a congruence subgroup. We define the space of quasimodular forms as :

$$\mathcal{QM}(\Gamma) := \mathcal{M}(\Gamma)[E_2].$$

Remark 2.14. This definition is not what is usually chosen in the litterature for the definition of quasimodular forms. However, it is equivalent to the usual definition.

Remark 2.15. E_2 can be seen as the fundamental "quasimodular" form. The way we usually prove a transformation formula for E_k for $k \geq 3$ is by interchanging sums, by, for example, the Poisson summation formula. The lack of absolute convergence of E_2 makes this a bit trickier for E_2. We can show that E_2 satisfies the following transformation formula,

$$E_2(\gamma\tau) = j(\gamma,\tau)^2 E_2(\tau) + \frac{6cj(\gamma,\tau)}{i\pi}$$

as proved in [12]

Example 2.16. The Fourier Expansion of E_2 is :

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \cdots$$

Remark 2.17. The space of quasimodular forms is the space of all derivatives of modular forms. Let us suppose we have some modular form E of weight k with respect to some Γ. Then,

$$E(\gamma\tau) = (c\tau + d)^k E(\tau)$$

Differentiating this equation gives :

$$\frac{1}{(c\tau+d)^2} E'(\gamma\tau) = (c\tau + d)^k E'(\tau) + kc(c\tau + d)^{k-1} E(\tau)$$

Hence,

$$E'(\gamma\tau) = (c\tau + d)^{k+2} E'(\tau) + kc(c\tau + d)^{k+1} E(\tau)$$

E' is "almost" modular of weight $k+2$.

2.5 Dirichlet characters and associated modular forms

In this subsection, let $N \in \mathbb{N}^*$.

Definition 2.18. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ be a group homomorphism. We define the *Dirichlet character* associated to χ, denoted $\tilde{\chi} : \mathbb{Z} \to \mathbb{C}$ by :

$$\tilde{\chi}(a) = \begin{cases} 0 \text{ if } \gcd(a,N) > 1 \\ \chi(a) \text{ if } \gcd(a,N) = 1 \end{cases} .$$

Remark 2.19. We will write χ for both the group homomorphism, and its extension to $\mathbb{Z}$.

Remark 2.20. As $|\,(\mathbb{Z}/N\mathbb{Z})^*\,| = \phi(n)$, where ϕ is Euler's totient function, Dirichlet characters have for values $\phi(N)$th roots of unity in $\mathbb{C}$.

Example 2.21. For N=3, there exists two morphisms $(\mathbb{Z}/3\mathbb{Z})^* \to \mathbb{C}$, one which is trivial, and the second which maps $[2]$ to -1.

Remark 2.22. If $(\mathbb{Z}/N\mathbb{Z})^*$ is cyclic, for example when N is a power of a prime, the choice of a Dirichlet character is just the choice of which $\phi(N)$th root of unity you send a generator to.

We will soon see the modular forms assoicated with certain Dirichlet characters. But first, just as the constant coefficients of certain modular forms uses the Riemann zeta function, we will need to define L-functions in their generality in order to find the constant coefficients of the following modular forms.

Definition 2.23. Let χ be a Dirichlet character modulo N. The associated *Dirichlet* $L-$*function* is :

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p\in\mathcal{P}} (1-\chi(p)p^{-s})^{-1}$$

Remark 2.24. We see that we can recover the usual Riemann zeta function as the Dirichlet L function associated to the trivial character. Moreover, just like the Riemann zeta function, there exist formulas that allow meromorphic continuation of an L function on the entire complex plane.

Definition 2.25. Let $k \geq 2$. Let ψ, ϕ be two Dirichlet characters modulo N. Define $E_k^{\psi,\phi}$ as the series with the following Fourier expansion :

$$E_k^{\psi,\phi}(\tau) = \delta(\psi)L(1-k,\phi) + 2\sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\phi}(n)q^n$$

where $q = e^{2i\pi\tau}$, $\delta(\psi) = 1$ if $\psi = 1_1$ and 0 otherwise.. Finally,

$$\sigma_{k-1}^{\psi,\phi}(n) = \sum_{\substack{m|n \\ m>0}} \psi(n/m)\phi(m)m^{k-1}.$$

Remark 2.26. These series are modular forms for $k \geq 3$ with respect to $\Gamma_1(N)$ ([3, Section 4.5]).

Eisenstein series of weight 1 also exist, and are defined as follows :

Definition 2.27. Let ψ, ϕ be two Dirichlet characters modulo N. Define $E_1^{\psi,\phi}$ as the series with the following Fourier expansion :

$$E_1^{\psi,\phi}(\tau) = \delta(\psi)L(0,\phi) + \delta(\phi)L(0,\psi) + 2\sum_{n=1}^{\infty} \sigma_0^{\psi,\phi}(n)q^n$$

where every term has the same meaning as the previous definition.

The usefuless of such series comes from the following theorem [3, Theorem 4.5.2]:

Theorem 2.28. *Let $k \geq 3$ and let $A_{n,k}$ be the set of all triples (ψ, ϕ, t) where ψ and ϕ are primitive Dirchlet characters modulo u and v such that $\psi\phi(-1) = (-1)^k$ and t is such that $tuv|N$. Define $E_k^{\psi,\phi,t}(\tau) := E_k^{\psi,\phi}(t\tau)$.*

Then the set

$$\{E_k^{\psi,\phi,t}|(\psi,\phi,t)\in A_{N,k}\}$$

is a basis of $\mathcal{E}_k(\Gamma_1(N))$.

2.6 Generators of spaces of modular forms

Now that we have seen generators of $\mathcal{E}_k(\Gamma_1(N))$ as a vector space, we could ask the following question : Can we find generators for the spaces $\mathcal{E}(\Gamma_1(N))$, or $\mathcal{M}(\Gamma_1(N))$ when they are viewed as algebras ? It was shown by the Deligne and Rapoport in 1973 in [2] that these spaces are finitely generated, and hence can be viewed as a quotient of $\mathbb{Q}[X_1,\cdots,X_N]$. More recently, in [9], Rustom proves that $M(\Gamma)$ is generated in weight at most 3 for $\Gamma=\Gamma_1(N)$.

In [4], Engel claims the following :

$$QM(\Gamma_1(3))=\mathbb{Q}[E_1^1,E_2,E_3^1]$$

where E_k^r is defined in the following way :

$$E_k^r=\theta_N\left((k-1)!\sum_{\substack{d\equiv r(N)\\ d\neq 0}}\left(\frac{-N}{2i\pi d}\right)^k+\sum_{n\geq 1}q^n\sum_{m|n}\mathrm{sgn}(m)m^{k-1}\zeta_N^{rm}\right)$$

where $\zeta_N=\exp(2i\pi/N)$, and $\theta_N=\zeta_N-\overline{\zeta_N}$;

Remark 2.29. Denote $\mathbb{1}_N$ the trivial character modulo N. For N=3,4,6, denote χ_N the unique non trivial character modulo N. Explicit computations of these series gives the following :

$$E_1^1(q)=-\frac{1}{2}-3q-3q^3-3q^4-6q^7-\cdots=-\frac{3}{2}E_1^{\mathbb{1}_1,\chi_3}(q)$$

$$E_3^1=\frac{1}{3}-3q+9q^2-3q^3-39q^4+72q^5+\cdots=-\frac{3}{2}E_3^{\mathbb{1}_1,\chi_3}(q).$$

Unable to find anywhere in the literature where such generators are given, I had to resort to trusting the author. A problem I encountered at this moment was that the programmation language I was using (Sage), didn't give me the same generators. It gave me 5 instead of 3. As it turns out, these 5 generators are equivalent to the 3 given by Engel, as both generate the space, but with different relations.

These are the 5 generators given by Sage (Sage doesn't consider modular forms of weight 1), and everyone of them can be written in the basis given by [4] above :

$$E2=1-24q-72q^2-96q^3-168q^4-\cdots=E_2$$

$$E2_0 = 1 + 12q + 36q^2 + 12q^3 + 84q^4 + \cdots = 4(E_1^1)^2$$
$$E3_0 = 1 + 54q^2 + 72q^3 + \cdots = 2E_3^1 - \frac{8}{3}(E_1^1)^3$$
$$E3_1 = q + 3q^2 + 9q^3 + 13q^4 + \cdots = -\frac{1}{9}E_3^1 - \frac{8}{27}(E_1^1)^3$$
$$E4_0 = 1 + 240q^3 + \cdots = \frac{16}{9}(E_1^1)^4 - \frac{16}{3}(E_1^1 E_3^1).$$

3 Counting ramified coverings of the sphere

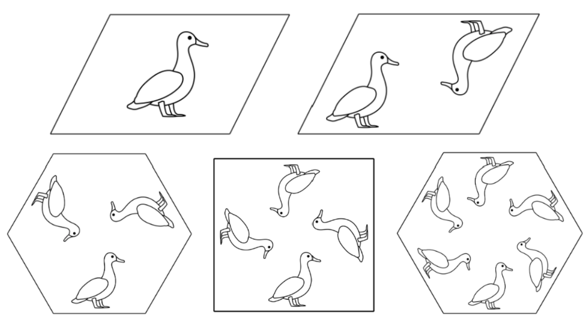
In this section, I'll mostly be following [4]. My goal is to compute generating series for the number of ramified coverings of spheres, with specific ramification data.

3.1 Tilings

There are seventeen *wallpaper groups*. Those are the groups of symmetries of biperiodic tilings of $\mathbb{C}$. Among those, 5 of them preserve orientation. These 5 groups are isomorphic to extensions of $\mathbb{Z}^2$ by a rotation group $< \zeta_N >$, for, respectively, $N = 1, 2, 3, 4, 6$. These groups are denoted G_N. We will write, from now on,

$$X_N = \mathbb{C}/G_N$$

Example 3.1. Here are the 5 tilings, whose group of symmetries is G_N :



credit : [4, Figure 1]

The counting of the number of branched covers has a combinatorial explanation. Roughly, the number of branched covers $\Sigma \to X_N$ with a specific ramification profile is the same as the number of tilings of Σ by the corresponding tile in example 3.1. For $N \neq 1$, these are topologically spheres.

Definition 3.2. Let $N = 1, 2, 3, 4, 6$. An N*-duck tilling* is a compact oriented surface formed from identifying pairs of edges of a collection of disjoint oriented N-duck tiles, as in example 3.1.

Let $\mathcal{T}$ be an N-duck tiling of a compact oriented surface Σ.

Definition 3.3. The *curvature* κ of a vertex of an $\mathcal{T}$ is

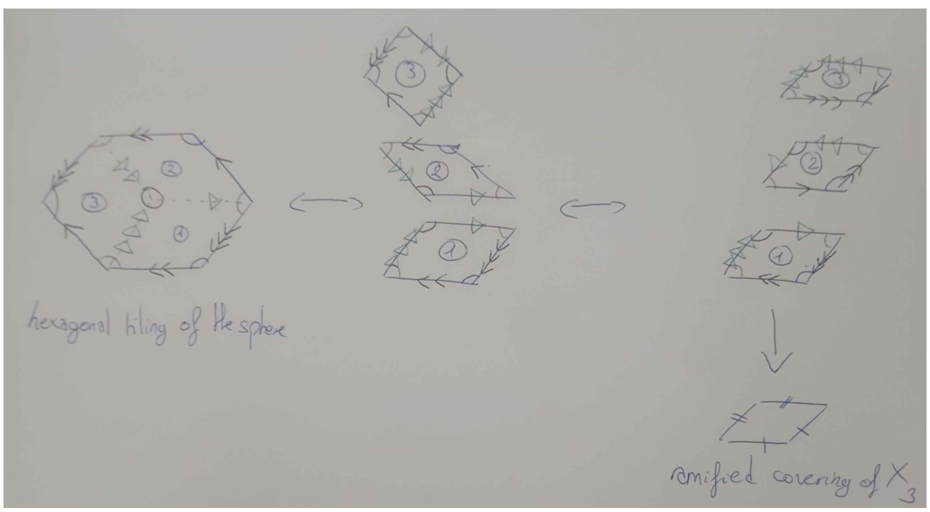
$$N - \frac{N-2}{N}k$$

where k is the number of tiles adjacent to the vertex.

Let $\mu = \mu_1 \geq \cdots \geq \mu_r$ be a partition of any integer. Define, for all$n \in \mathbb{N}$, $\mu + n := \mu_1 + n \geq \cdots \geq \mu_r + n$ The following proposition relates tilings and coverings for the case $N = 3$. Its proof can be found in [4, Proposition 2.4].

Proposition 3.4. *The* 3*-duck tiled surfaces* Σ *with* d *tiles and curvature* $3 - \mu$ *are in bijection with degree* $3d$ *covers* $\sigma : \Sigma \to \mathbb{P}^1$ *ramified over three points with cycle type* $(3^d, 3^d, (\mu, 3^{d-|\mu|/3}))$.

Example 3.5. The following picture is the example of the equivalence between the covering with ramification $(3,3,3)$ of X_3 with the tiling of the sphere.



3.2 Generating function of number of branched covers

Fix n points $p_1, \cdots p_n$ on $\mathbb{P}^1$, and a starting point $x \in \mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}$.

Now, let γ_i be a closed loop around p_i based in x. Then, the fundamental group of $\mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}$ is

$$\pi_1(\mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}, x) =< \gamma_1, \cdots, \gamma_n | \ \gamma_1 \cdots \gamma_n = 1 > .$$

Remark 3.6. This can be computed in the following way : a punctured sphere is homeomorphic to the plane, and the $n-1$-punctured plane deformation retracts to a bouquet of $n-1$ circles. Because the bouquet of $n-1$ circles has for fundamental group the free group on $n-1$ generators, the kth generator being a clockwise 1-loop around the kth circle. Bringing these back to the sphere gives $n-1$ generators, and the last one can be obatined as the inverse of the product of the $n-1$ others.

Definition 3.7. Let $\sigma : X \to \mathbb{P}^1$ be a covering, which is ramified in $p_1, \cdots p_n$. Fix $y \in \sigma^{-1}(x)$. If $\gamma \in \pi_1(\mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}, x)$, then, by the lifting criterion of coverings, γ can be lifted to X, denoted $\tilde{\gamma}_y$, such that $\tilde{\gamma}_y(0) = y$ and $\tilde{\gamma}_y(1) \in \sigma^{-1}(x)$. The action

$$\pi_1(\mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}, x) \times \sigma^{-1}(x) = \sigma^{-1}(x)$$
$$(\gamma, y) \mapsto \tilde{\gamma}_y(1)$$

is said to be the action of *monodromy* of $\pi_1(\mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}, x)$ on $\sigma^{-1}(x)$.

Example 3.8. The covering $\exp(2i\pi\cdot) : \mathbb{R} \to \mathbb{S}^1$, where $x = 1$ has for monodromy the maps which takes (γ_n, k) where $k, n \in \mathbb{Z}$ and γ_n loops n times around the circle and maps it to $k + n$

If we have a finite cover $\sigma : X \to \mathbb{P}^1$, fix a labeling of the fiber over x by $\{1, \cdots, d\}$. We say that σ has ramification profile η_i at p_i if the cycle type of the action of monodromy of γ_i on the fiber of x is of cycle type η_i

Definition 3.9. Let $d \in \mathbb{N}^*$. Define the *Hurwitz number* as

$$H_d(\eta_1, \cdots \eta_n) = \sum_{\sigma} \frac{1}{|Aut(\sigma)|}$$

where the sum is over all branched covers $\sigma : X \to \mathbb{P}^1$ of degree d with ramification profiles η_i over p_i where η_i is in a partition of d.

Definition 3.10. We also define the generating series of these H_d:

$$\mathcal{C}(\eta_1, \cdots \eta_n)(q) = \sum_{d \geq 0} H_d(\eta_1, \cdots \eta_n) q^d.$$

Denote C_{η_i} the conjugacy class of S_d of elements with cycle type η_i. Each cover σ defines, by monodromy, elements $(s_1, \cdots, s_n) \in C_{\eta_1} \times \cdots \times C_{\eta_n}$ such that $s_1 \cdots s_n = 1$. Conversly, each $s_1, \cdots, s_n \in C_{\eta_1} \times \cdots \times C_{\eta_n}$ such that $s_1 \cdots s_n = 1$ defines a cover $\sigma : X \to \mathbb{P}^1$. Indeed, let $p : \tilde{X} \to \mathbb{P}^1 \setminus \{p_1, \cdots p_n\}$ be the universal cover. Then, let $F = \{1, \cdots, d\}$. Quotient $\tilde{X} \times F$ by $([\gamma], y) \sim ([\gamma_i \gamma], s_i(y))$ for all $i \in \{1, \cdots, n\}$. The resulting cover can be extended to a ramified cover by Riemann's extension theorem on compact Riemann surfaces, and has wanted ramification data.

Moreover, two coverings X_1 and X_2 are isomorphic if and only if the induced bijection on fibers $f : F_1 \to F_2$ has the following property : $f(\tilde{\gamma}_y(1)) = \tilde{\gamma}_{f(y)}(1)$. In turn, this shows that d-sheeted coverings of $\mathbb{P}^1$ with ramification $s_1, \cdots, s_d$ are in bijection with

$$\{\sigma \in \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}, x), S_d)\ \forall i \in \{1, \cdots, n\},\ \sigma(\gamma_i) \in C_{\eta_i}\}/S_d$$

where the action is by conjugation. Denote the above set A/S_d.
Then,

$$\begin{aligned}
H_d(\eta_1, \cdots \eta_n) &= \sum_{\sigma \in A/S_d} \frac{1}{|Aut(\sigma)|} \\
&= \sum_{\sigma \in A/S_d} \frac{|orb(\sigma)|}{d!} \\
&= \frac{|A|}{d!}
\end{aligned}$$

where the second equality is due to orbit-stabilizer.

Now, $|A| = |\{(s_1, \cdots, s_n) \in C_{\eta_1} \times \cdots \times C_{\eta_n} |\ s_1 \cdots s_n = 1\}|$. Hence, using Frobenius' formula A.10 for $g = 0$, we have the following :

Proposition 3.11.

$$H_d(\eta_1, \cdots \eta_n) = \frac{|C_{\eta_1}| \cdots |C_{\eta_n}|}{(d!)^2} \sum_{\chi} \frac{\chi(C_{\eta_1}) \cdots \chi(C_{\eta_n})}{\chi(1)^{n-2}}$$

I would like to count connected covers of X_N, but this series counts all covers. To solve this problem, we introduce the following :

Definition 3.12. Let $\mathcal{C}^{\bullet}$ denote the generating function for coverings without unramified components. That is,

$$\mathcal{C}^{\bullet}(\eta_1, \cdots \eta_n)(q) = \sum_{d \geq 0} H_d^{\bullet}(\eta_1, \cdots \eta_n) q^d.$$

Lemma 3.13.

$$\mathcal{C}^{\bullet}(\eta_1, \cdots \eta_n)(q) = \frac{\mathcal{C}(\eta_1, \cdots \eta_n)(q)}{\mathcal{C}()(q)}$$

Proof. By multplying both sides by $\mathcal{C}()(q)$, and comparing term by term, this is equivalent to asking that

$$H_d(\eta_1, \cdots, \eta_n) = \sum_{k=0}^{d} H_k^{\bullet}(\eta_1, \cdots, \eta_n) H_{d-k}().$$

Having a degree d cover is the same as choosing k elements of the fiber such that on $d - k$ of them is a cover without unramified components, and, on the k remaining, a cover without any ramification.

$\square$

Now, we need to introduce a third series :

Definition 3.14. Let $\mathcal{C}^\bullet$ denote the generating function for connected coverings. That is,

$$\mathcal{C}^\circ(\eta_1, \cdots \eta_n)(q) = \sum_{d \geq 0} H_d^\circ(\eta_1, \cdots \eta_n) q^d.$$

Remark 3.15. To pass from covers with each component ramified to connected covers, there is a solution by Mobius inversion, see, for example, [5, Section 2.3].

The following theorem si the fundamental result of the article [4] :

Theorem 3.16. *The generating series*

$$\mathcal{C}^\circ(\eta_1, \cdots \eta_n)(q)$$

is quasimodular.

Guided by proposition 3.4, in the following, I'll only be interested in the case where $\mathbf{N} = \mathbf{3}$, $n = 3$, d is a multiple of 3 $\eta_1 = \eta_2 = (3, 3 \cdots 3)$, μ is a permutation of $3k \leq d$, and $\eta_3 = (\mu, 3 \cdots 3)$.

My goal is to compute as many terms as I can of the series counting connected covers:

$$\mathcal{C}^\circ(\eta_1, \eta_2, \eta_3)(q).$$

Doing so proves to be a problem, because, one needs to access characters of S_{3d}, which take a long time to compute. Direct computation using character tables quickly becomes too time-consuming. To overcome this, I implemented formulas to give me the character table of S_{3d}. This proves to be quite inneficient, as it necessitates computing all characters of S_{3d}, for all partitions of $3d$, whereas I just need all characters for the partitions η_1, η_2, and η_3. The second idea was then to compute only characters for the partitions I needed. I used the Murnaghan-Nakayama formula and formalism for this. Alas, this also proved to be inneficient. The last idea, and the one I went with, was to implement two formulas, one which gives values for $|\chi_\lambda(3, \cdots, 3)|$ (Theorem 3.39), another which gives values for $\chi_\lambda(1)$, and to use Murnaghan-Nakayama on η_3, up until μ is finished, and then compute the rest by the preceding formula. Unfortunately, Theorem 3.39 doesn't give the sign of the character, so I used Proposition 3.42 to compute the sign. Analysis of the performances of this algorithm can be found in subsection 3.4.

The following theorem si the fundamental result of the article [4] :

Theorem 3.17. *The generating series*

$$\mathcal{C}^\circ(\eta_1, \eta_2, \eta_3)(q)$$

is quasimodular for $\Gamma_1(3)$.

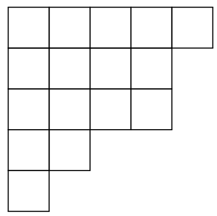
Remark 3.18. The quasimodularity is useful because it allows us to write $\mathcal{C}^\circ(\eta_1, \eta_2, \eta_3)(q)$ in terms of known series, which allows us to find the asymptotic of these series as $q \to 1$, which, in turn, allows us to compute Masur-Veech volumes.

3.3 Murnaghan-Nakayama

Fix some $n \in \mathbb{N}$. Fix also two partitions λ and μ of n. The Murnaghan-Nakayama rule is an algorithm which computes the value of $\chi_\lambda(\mu)$. For this, I'll briefly recall the formalism of Young diagrams.

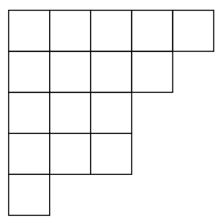
Definition 3.19. If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq 0)$, then the *Young diagram* associated to λ is the collection of unit boxes, whose ith row has length λ_i.

Example 3.20. The Young Diagram associated to $(5, 4, 4, 2, 1)$ is the following :



Definition 3.21. The *transpose partition* of λ, denoted λ' is the partition obtained by exchanging rows and columns in the Young diagram of λ.

Example 3.22. For $\lambda = (5, 4, 4, 2, 1)$, we have $\lambda' = (5, 4, 3, 3, 1)$

$\lambda' =$ 

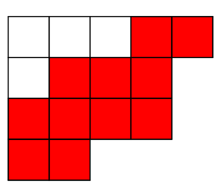
Define $p_i := \lambda_i - i$ and $q_i = \lambda'_i - i$. The reason for using such notations will be geometrically apparent later. With these definitions, we have the following theorem : ([8, eq 2.7])

Theorem 3.23.

$$\dim \lambda = \frac{\prod_{i<j}(p_i - p_j)(q_i - q_j)}{\prod_{i,j}(p_i + q_j + 1)\prod_i(p_i!q_i!)}$$

Definition 3.24. Let λ and μ be two partitions such that $\mu_i \leq \lambda_i$, $\forall i \in \{1, \cdots, n\}$. A *Skew Young Diagram* λ/μ consists of the cells in the Young diagram of λ that don't belong in the Young Diagram of μ.

Example 3.25. The Skew Young Diagram for $\lambda = (5, 4, 4, 2)$ and $\mu = (3, 1)$ is highlighted in red :

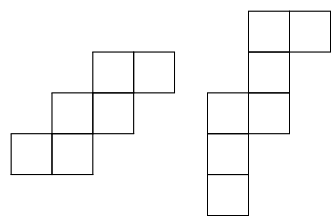


Definition 3.26. A strip is a connected Skew Young Diagram that does not contain any 2x2 block.

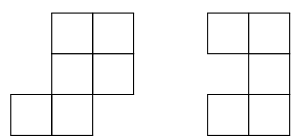
Definition 3.27. The *height* of a strip is the number of rows in the strip, minus one.

Definition 3.28. The *size* of a strip is the number of boxes in it.

For example, the following are strips of length 2 and 4 :



And the following are *not* strips :



Definition 3.29. A μ-strip decomposition of λ is a way of recovering the Young diagram of λ by succesively adjoing strips of size $\mu_1, \mu_2, \cdots$.

Example 3.30. A $(5, 3, 2, 2, 2, 1)$-decomposition of $(5, 4, 4, 2)$:

1	1	1	3	6
1	2	2	3	
1	2	5	5	
4	4			

Here, the number 1 represents the strip of size 5, the number 2 the strip of size 3, etc...

Definition 3.31. The *height* of a μ strip decomposition is the sum of the heights of the μ_i strips.

Example 3.32. The above μ-strip decomposition has height $2+1+1+0+0+0 = 4$.

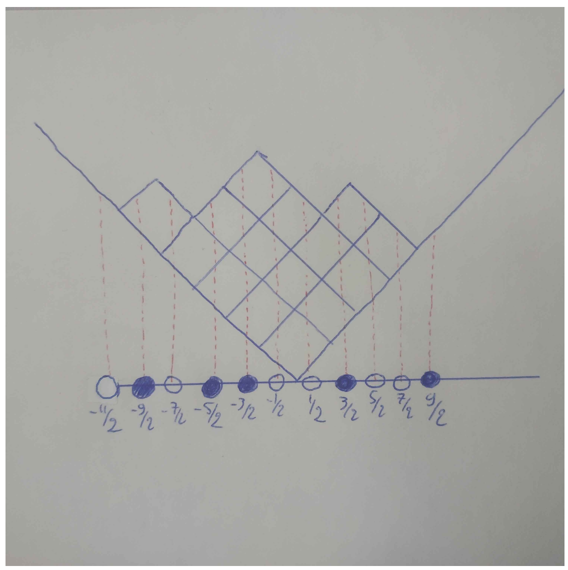
The Murnaghan-Nakayama algorithm is an algorithm which takes in the Young diagram of λ, and the partition μ, and removes so μ *strips* from the diagram, and computes the value of the characters of the symmetric group $\chi_\lambda(\mu)$. This can be found in [11, Theorem 7.17.3] :

Theorem 3.33. *The non-recursive Murnaghan Nakayama rule is the following:*

$$\chi_\lambda(\mu) = \sum_{\{\eta_i\}} (-1)^{ht\{\eta_i\}}$$

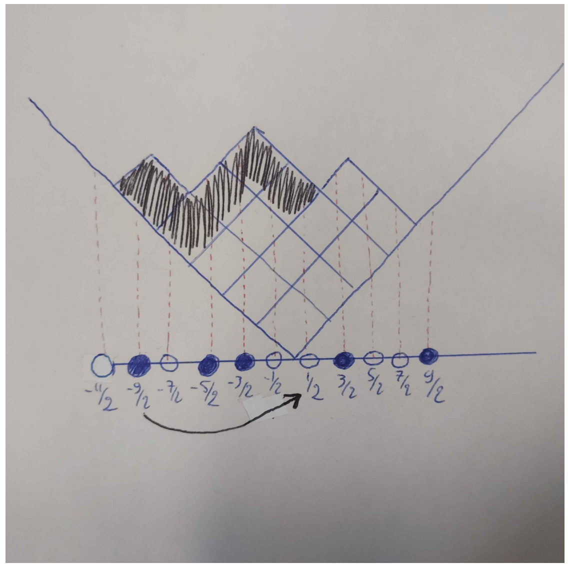
where the sum runs over all μ-strip decompositions of λ.

Now we'll see a way of visualizing Young diagrams, which makes computing strip decompositions more visual. Take a Young Diagram, rotate it by 90 so that the top left box is at the bottom. Place beads under it at every half integers. Color the beads where the boundary of the Young Diagram is going up. This coincides with $p_i + 1/2$. Here is an example for $(5, 4, 4, 2)$:



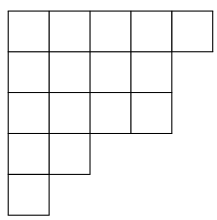
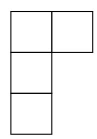
This view is useful, because removie a strip of size t is the same as moving a full bead where the strip starts t to the right. Hence, Murnaghan-Nakayama can be interpreted in terms of moving beads to the right, which makes it easier to understand.

Example 3.34. In the following drawing, removing the blackened strip moves the bead at -9/2 5 to the right, up to 1/2 :



Definition 3.35. The *t-core* of λ is the partition obtained from λ after removing t strips as much as possible.

Example 3.36. The 3-core of $\lambda = (5,4,4,2,1)$ is $(2,1,1)$

$\lambda =$  core $=$ 

Definition 3.37. Let λ be a partition, and $p_1, \cdots p_k$ the positions at the half integers of its beads in the Maya diagram. Let $t \in \mathbb{N}$. Consider the t partitions obtained by taking, for the ith partition, only the beads that are congruent to $i + 1/2$ modulo p. The set of all these partitions is called the *t-quotient* of λ.

Example 3.38. The 2 quotient of the example 3.3 is the one made of two partitions, one which has beads at $-9/2$, $-5/2$, and $3/2$, the other which has a single bead at $-3/2$.

We have the following formula, which is a corrollary of Murnaghan-Nakayama:

Theorem 3.39 (Formula for $(p,p,\cdots,p)$)**.** *Let $p \in \mathbb{N}^*$, and λ a partition of $n = pk$. Let μ be the p-core of λ, and $\lambda^1, \cdots, \lambda^p$ its p-quotient. Then,*

$$\chi_\lambda(p,\cdots,p) = \begin{cases} 0 & \textit{if } \mu \neq 0 \\ \binom{|\lambda|/p}{|\lambda^1|,|\lambda^2|,\cdots,|\lambda^p|} \prod_{i=1}^p \dim \lambda^i & \textit{if } \mu = 0 \end{cases}$$

where
$$\binom{|\lambda|/p}{|\lambda^1|, |\lambda^2|, \cdots, |\lambda^p|} = \frac{(|\lambda|/p)!}{|\lambda^1|! \cdots |\lambda^p|!}.$$

Proof. If the μ-core of λ is non-zero, then there exists no μ-strip decompositions of λ, and so $\chi_\lambda(p, \cdots, p) = 0$ by Murnaghan-Nakayama. Now, let us suppose the μ-core of λ is trivial. Adding a p-strip is the same as adding a one cell to the corresponding p-quotient. But, by Murnaghan-Nakayama, counting the number of ways of adding one-cells to a component of the p-quotient λ^i is the same as computing the dimension $\dim \lambda^i$. Moreover, as Murnaghan-Nakayama sums over all possible μ strip decompositions, we have to take into account the order in which the adding of the one cells in the p-quotient is done. There are p partitions in the p-quotient, and couting the number of ways of adding a one cell to each element of the p-quotient is the same as asking how many ways there are of counting the number of ways of partitioning $\{1, \cdots, k\}$ in sets of size $|\lambda^1|, \cdots |\lambda^p|$. This is exactly the multinomial coefficient
$$\binom{|\lambda|/p}{|\lambda^1|, |\lambda^2|, \cdots, |\lambda^p|}.$$
Which gives us the desired result. □

In order to find the sign of $|\chi_\lambda(p, \cdots, p)|$, I use the following proposition 3.42, which needs the following remark:

Remark 3.40. The height of a strip is the same as the number of beads that it would jump over when removing the strip in the Maya diagram. Indeed, each bead placed between the starting position and the ending position of the moved bead corresponds to adding +1 height to the strip, as seen in the two drawings in 3.3 and 3.34.

Example 3.41. In the example 3.3, moving the bead at $-5/2$ to $1/2$ has a number of jumps 1. Moving it from $-5/2$ to $5/2$ has number of jumps 2. This is the same as its height.

I have found the following nowhere in the litterature, so I decided to include it here. It probably is somewhere. It is a direct corollary of Murnaghan Nakayama.

Proposition 3.42. *The sign of* $\chi_\lambda(p, \cdots, p)$ *is*
$$(-1)^{\sum heights}$$
where the sum is over the removal of all p-strips, or equivalently, moving all possible beads p to the right. This sum depends on how you remove strips, but its parity does not.

Proof. Suppose you have two players, A and B, who both choose different strategies to move all the beads p to the right. Choose two beads, labeled x and y. After the two players have moved all the beads as much as they can, the two beads x and y are at the same positions, whether it is by the strategy of A or that of B. So, if A had x jump over y k more times than B, then, mechanically,

y would have jumped over x k more times also by A. This means that the sum of heights, wether by the strategy of A or that of B, has the same parity.

This proves that the parity of the sum of heights does not depend on which way you remove p-strips. This means that all $\mu = (p, \cdots, p)$ strips decompositions of λ have the same parity of the sum of heights of removed strips. But, Murnaghan-Nakayama states that

$$\chi_\lambda(\mu) = \sum_{\{\eta_i\}} (-1)^{\mathrm{ht}\{\eta_i\}}$$

Each of the terms in this sum then have the same sign, which gives the desired result. □

Remark 3.43. This does not work if the permutation is not of the form $p, p, \cdots, p$. Take, for instance, $\lambda = (2, 2)$, $\mu = (2, 1, 1)$. There are two different ways of removing 2 from $(2, 2)$, one of which has height 1, and the other height 0. Hence, the above proposition depends on how the beads are moved to the right.

Example 3.44. The computation of the sign of $\chi_\lambda(p, \cdots, p)$ is done in the following code, in the appendix : B.2.

3.4 Algorithmic performance

To compute these series $H^\bullet(\mu)$, I used the algorithm I described at the end of subsection 3.2. The two main obstacles to performant computation are the following :

1. The number of terms in $H^\bullet(\mu)$.
2. The number of parts in μ.

1. The dth term in the series $H^\bullet(\mu)$ requires computing a sum over all characters of the symmetric group S_{3d}. As the number of characters is equal to the number of conjugation classes of S_{3d}, it equals $p(3d)$, the number of partitions of $3d$, whose first terms are

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, \ldots.$$

(see, for instance : A000041).

The following theorem, proved by Hardy and Ramanujan, gives an asymptotic of $p(n)$:

Theorem 3.45.

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Example 3.46. For small μ (of size ≤ 4), I am able to go to the 15th term in about 3-4 minutes. This means computing all characters of S_{45} at three different values, and the above formula tells us that there are about $p(45) \approx 95316$ of them.

2. My algorithm runs Murnaghan-Nakayama on μ. Hence, the more parts μ has, the more computationally expensive it becomes. If I go up to the 10th term of the series, I can compute them for all μ under size 6 without any problems. But, computing the 7th term for μ of size 9 took me about an hour and a half. This is still way faster than asking Sage to compute the entire table. I only compute 3 columns of the character table, instead of all of them.

My code probably isn't as optimised as it could be, and most of it can be found in Appendix B.

3.5 Numerical Values

In this section, I will post the values I have found for $H^\bullet(\mu)$. A list $[a_0, a_1, \cdots, a_k]$ in the second column corresponds to a modular form $a_0 + a_1 q + a_2 q^2 + \cdots + a_k q^k + \cdots$. I will only show non zero $H^\bullet(\mu)$.

μ	$H^\bullet(\mu)$	In the basis E_k^i
$[1,1,1]$	$[0,\frac{1}{3},0,\frac{1}{3},\frac{1}{3},0,0]$	$-\frac{1}{18}-\frac{1}{9}(E_1^1)$
$[5,1]$	$[0,0,1,1,2,2,4]$	$\frac{5}{72}-\frac{1}{72}(E_2)+\frac{1}{9}(E_1^1)$
$[2,2,1,1]$	$[0,0,\frac{1}{2},1,1,1,\frac{7}{2}]$	$\frac{1}{18}-\frac{1}{72}(E_2)+\frac{1}{18}(E_1^1)-\frac{1}{18}(E_1^1)^2$
$[1,1,1,1,1,1]$	$[0,0,\frac{1}{18},0,\frac{1}{9},\frac{1}{9},\frac{1}{18}]$	$\frac{1}{648}+\frac{1}{162}(E_1^1)+\frac{1}{162}(E_1^1)^2$
$[7,1,1]$	$[0,0,0,2,4,17,15,33]$	$-\frac{343}{1296}+\frac{5}{54}(E_3^1)+\frac{7}{144}(E_2)-\frac{91}{216}(E_1^1)+\frac{1}{24}(E_1^1)(E_2)+\frac{2}{27}(E_1^1)^2+\frac{5}{27}(E_1^1)^3$
$[5,2,2]$	$[0,0,0,1,1,5,3,9]$	$-\frac{5}{162}-\frac{1}{54}(E_3^1)+\frac{1}{72}(E_2)-\frac{1}{54}(E_1^1)+\frac{1}{54}(E_1^1)^2-\frac{2}{27}(E_1^1)^3$
$[5,1,1,1,1]$	$[0,0,0,\frac{1}{3},\frac{1}{3},2,\frac{10}{3},\frac{10}{3}]$	$-\frac{55}{1296}+\frac{1}{27}(E_3^1)+\frac{1}{1296}(E_2)-\frac{23}{216}(E_1^1)+\frac{7}{648}(E_1^1)(E_2)-\frac{5}{162}(E_1^1)^2+\frac{7}{81}(E_1^1)^3$
$[4,4,1]$	$[0,0,0,2,3,8,8,20]$	$-\frac{4}{81}-\frac{1}{54}(E_3^1)+\frac{1}{54}(E_2)-\frac{2}{27}(E_1^1)+\frac{1}{108}(E_1^1)(E_2)-\frac{1}{27}(E_1^1)^3$
$[4,2,1,1,1]$	$[0,0,0,0,2,4,4,12]$	$-\frac{8}{81}+\frac{1}{27}(E_3^1)+\frac{1}{54}(E_2)-\frac{4}{27}(E_1^1)+\frac{1}{54}(E_1^1)(E_2)+\frac{2}{27}(E_1^1)^2+\frac{10}{81}(E_1^1)^3$
$[2,2,2,2,1]$	$[0,0,0,0,0,1,0,1]$	$-\frac{5}{324}+\frac{1}{216}(E_2)-\frac{1}{108}(E_1^1)+\frac{1}{54}(E_1^1)^2-\frac{1}{81}(E_1^1)^3$
$[2,2,1,1,1,1,1]$	$[0,0,0,\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{5}{6},\frac{11}{6}]$	$-\frac{1}{324}+\frac{1}{1296}(E_2)-\frac{1}{108}(E_1^1)+\frac{1}{648}(E_1^1)(E_2)-\frac{1}{324}(E_1^1)^2+\frac{1}{162}(E_1^1)^3$
$[1,1,1,1,1,1,1,1,1]$	$[0,0,0,\frac{1}{162},0,\frac{1}{54},\frac{1}{54},\frac{1}{54}]$	$-\frac{1}{34992}-\frac{1}{5832}(E_1^1)-\frac{1}{2916}(E_1^1)^2-\frac{1}{4374}(E_1^1)^3$
$[11,1]$	$[0,0,0,0,24,92,232,463,835,1202,1945,2679]$	$\frac{94501}{51840}-\frac{1}{6}(E_3^1)-\frac{3025}{5184}(E_2)+\frac{275}{10368}(E_2)^2+\frac{1573}{648}(E_1^1)-\frac{37}{60}(E_1^1)(E_3^1)-\frac{143}{648}(E_1^1)(E_2)-\frac{44}{81}(E_1^1)^2-\frac{1}{3}(E_1^1)^3-\frac{37}{20}(E_1^1)^4$

μ	$H^\bullet(\mu)$	In the basis E_k^i
$[10,2]$	$[0,0,0,0,10,40,90,200,300,480,780,1040]$	$\frac{125}{243}+\frac{10}{81}(E_3^1)-\frac{25}{108}(E_2)+\frac{5}{432}(E_2)^2+\frac{100}{243}(E_1^1)-\frac{25}{81}(E_1^1)(E_3^1)-\frac{20}{81}(E_1^1)^2+\frac{40}{81}(E_1^1)^3-\frac{25}{27}(E_1^1)^4$
$[8,4]$	$[0,0,0,0,8,32,64,128,224,320,512,704]$	$\frac{256}{1215}+\frac{4}{81}(E_3^1)-\frac{8}{81}(E_2)+\frac{1}{162}(E_2)^2+\frac{64}{243}(E_1^1)-\frac{56}{405}(E_1^1)(E_3^1)-\frac{2}{81}(E_1^1)(E_2)+\frac{8}{81}(E_1^1)^3-\frac{56}{135}(E_1^1)^4$
$[8,2,1,1]$	$[0,0,0,0,8,32,80,160,336,448,720,1024]$	$\frac{256}{243}-\frac{16}{81}(E_3^1)-\frac{8}{27}(E_2)+\frac{1}{81}(E_2)^2+\frac{320}{243}(E_1^1)-\frac{8}{81}(E_1^1)(E_3^1)-\frac{4}{27}(E_1^1)(E_2)-\frac{64}{81}(E_1^1)^2+\frac{4}{81}(E_1^1)^2(E_2)-\frac{16}{27}(E_1^1)^3-\frac{32}{81}(E_1^1)^4$
$[7,5]$	$[0,0,0,0,4,22,38,101,147,236,369,527]$	$\frac{8575}{31104}-\frac{1}{162}(E_3^1)-\frac{175}{1728}(E_2)+\frac{7}{1152}(E_2)^2+\frac{665}{1944}(E_1^1)-\frac{55}{324}(E_1^1)(E_3^1)-\frac{1}{72}(E_1^1)(E_2)-\frac{2}{81}(E_1^1)^2+\frac{1}{27}(E_1^1)^3-\frac{55}{108}(E_1^1)^4$
$[7,2,2,1]$	$[0,0,0,0,4,15,38,100,141,241,390,519]$	$\frac{343}{648}-\frac{161}{864}(E_2)+\frac{7}{864}(E_2)^2+\frac{287}{648}(E_1^1)-\frac{4}{27}(E_1^1)(E_3^1)-\frac{1}{72}(E_1^1)(E_2)-\frac{101}{216}(E_1^1)^2+\frac{5}{216}(E_1^1)^2(E_2)+\frac{13}{81}(E_1^1)^3-\frac{13}{27}(E_1^1)^4$
$[7,1,1,1,1,1]$	$[0,0,0,0,\frac{2}{3},\frac{4}{3},\frac{19}{3},9,23,26,44,\frac{191}{3}]$	$\frac{5831}{116640}-\frac{19}{486}(E_3^1)-\frac{7}{2592}(E_2)+\frac{112}{729}(E_1^1)-\frac{23}{1215}(E_1^1)(E_3^1)-\frac{1}{54}(E_1^1)(E_2)+\frac{167}{1944}(E_1^1)^2-\frac{5}{648}(E_1^1)^2(E_2)-\frac{26}{243}(E_1^1)^3-\frac{2}{45}(E_1^1)^4$
$[5,5,1,1]$	$[0,0,0,0,6,9,37,56,121,163,259,329]$	$\frac{8875}{31104}-\frac{8}{81}(E_3^1)-\frac{325}{5184}(E_2)+\frac{1}{384}(E_2)^2+\frac{475}{972}(E_1^1)-\frac{7}{324}(E_1^1)(E_3^1)-\frac{19}{324}(E_1^1)(E_2)-\frac{5}{162}(E_1^1)^2+\frac{1}{162}(E_1^1)^2(E_2)-\frac{2}{9}(E_1^1)^3-\frac{13}{324}(E_1^1)^4$
$[5,4,2,1]$	$[0,0,0,0,4,28,72,132,236,384,560,792]$	$\frac{40}{81}+\frac{1}{27}(E_3^1)-\frac{31}{162}(E_2)+\frac{1}{108}(E_2)^2+\frac{44}{81}(E_1^1)-\frac{2}{9}(E_1^1)(E_3^1)-\frac{7}{162}(E_1^1)(E_2)-\frac{22}{81}(E_1^1)^2+\frac{1}{81}(E_1^1)^2(E_2)+\frac{2}{27}(E_1^1)^3-\frac{16}{27}(E_1^1)^4$
$[5,2,2,1,1,1]$	$[0,0,0,0,\frac{5}{6},\frac{35}{6},14,\frac{62}{3},\frac{113}{2},\frac{159}{2},102,\frac{955}{6}]$	$\frac{1865}{11664}-\frac{11}{243}(E_3^1)-\frac{71}{1728}(E_2)+\frac{11}{5184}(E_2)^2+\frac{2689}{11664}(E_1^1)+\frac{1}{486}(E_1^1)(E_3^1)-\frac{43}{1296}(E_1^1)(E_2)-\frac{539}{3888}(E_1^1)^2+\frac{7}{432}(E_1^1)^2(E_2)-\frac{89}{486}(E_1^1)^3+\frac{7}{486}(E_1^1)^4$

μ	$H^\bullet(\mu)$	In the basis E^i_k
[13,1,1]	[0,0,0,0,0,120,588,1794,4110,8520,14664, 14664,26154,38649,60891,85308,123762,158760]	$-\frac{4084223}{311040}+\frac{4303}{1296}(E^1_3)+\frac{37349}{10368}(E_2)-\frac{23}{144}(E_2)(E^1_3)$ $-\frac{3211}{20736}(E_2)^2-\frac{3126331}{155520}(E^1_1)+\frac{4999}{3240}(E^1_1)(E^1_3)+\cdots$

A Frobenius Formula

Let me recall a bit of representation theory of finite groups here. Most of my knowledge comes from [10]. The last result about Frobenius' formula are in [7]. We let G be a finite group in this section.

Definition A.1. A *Representation* of G is a tuple (V, ρ) where V is a finite dimensional complex vector space, and $\rho : G \to GL(V)$ is a group homomorphism.

Once a structure is defined, the next step is always to define what morphisms between these structure are, and we'll see (Shur's Lemma) that morphisms between representations are really few.

Definition A.2. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of G. A *morphism of representations* between V_1 and V_2 is a linear map $V_1 \to V_2$ such that, for all $g \in G$, the following diagram commutes

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(g)"] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\phi"] & V_2 \end{tikzcd}$$

Example A.3. If G is a group, let $\mathbb{C}[G]$ be its *Group Algebra*, that is, the vector space whose basis is indexed by elements of G, denoted e_g for $g \in G$. We have a representation $(\mathbb{C}[G], \rho)$ where $\rho(g)(e_k) = e_{gk}$ This representation is very useful because it has very nice properties that we'll see later.

Remark A.4. From now on, I'll drop the ρ when talking about the representation when it is obvious in the context.

Definition A.5. Let V be a representation of G. A $sub-representation$ of V is a sub vector space $W \subset V$ such that W is invariant under the induced action of G. That is, for all $g \in G$ and $w \in W, \rho(g)(w) \in W$.

Definition A.6. An *Irreducible representation* V is a representation that has no non-trivial sub-representations.

Once we have introduced irreducible representations, the natural following question arises : Can every representation of G be decomposed as a sum of irreducible representations ? The answer to this question is : YES. It suffices to show that a subrepresentation of G has a supplementary representation. The key idea is to create a scalar product "invariant under G", such that if W is stable under G, the orthogonal of W under this scalar product is also invariant under G. This is summed up in the following theorem :

Theorem A.7. *Let V be a representaiton of G. Then, there exists irreducible representations V_i of G such that*

$$V = \bigoplus_{i=1}^{k} V_i$$

A particular case of this decomposition is the case when $V = \mathbb{C}[G]$, in which we have the following theorem.

Theorem A.8. *Let $(V_i)_{i=1}^k$ be the irreducibles representations of G. We have the following decomposition :*

$$\mathbb{C}[G] = \bigoplus_{i=1}^{k} \underbrace{V_i \oplus \cdots \oplus V_i}_{k\ times}.$$

The following very important lemma now describes how rare morphisms of representations are :

Lemma A.9 (Shur's Lemma)**.** *Let V, W be two irreducible representations of G. Then,*

$$\mathrm{Hom}_G(V, W) = \begin{cases} \{0\}, & \text{if } V \ncong W \\ \mathbb{C}Id, & \text{if } V \cong W \end{cases}$$

All of this allows us to state the following theorem, whose proof I shall give here. It computes the number

$$\mathcal{N}_g(C_1, \cdots C_k) = |\{a_1, \cdots, a_g, b_1, \cdots, b_g, c_1, \cdots, c_k \in \\ G^{2g} \times C_1 \times \cdots \times C_k | [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_k = 1\}|.$$

Theorem A.10 (Frobenius Formula)**.** *Let $g \geq 0$. Let $C_1, \cdots, C_k$ be conjugacy classes in G. Then,*

$$\mathcal{N}_g(C_1, \cdots C_k) = |G|^{2g-1} |C_1| \cdots |C_k| \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}}$$

where the sum is over all irreducible characters of G

Remark A.11. We will only be using this theorem for $G = S_n$.

Proof. Let $a_1, \cdots, a_g, b_1, \cdots, b_g, c_1, \cdots, c_k \in G^{2g} \times C_1 \times \cdots \times C_k$ such that $[a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_k = 1$ Because $[a_i, b_i] = a_i (b_i a_i b_i^{-1})^{-1}$, counting the number of these a_i and b_i that satisfy the given condition is almost the same as counting the number of $a_i \in G$ and $a_i' \in C_{a_i}^{-1}$ which multiplied all together and by the c_i's, gives 1. The only thing that needs to be taken care of is the number of $b \in G$ such that if a_i and a_i' are in the same conjugacy class, $b a_i b^{-1} = a_i'$.By the orbit stabilizer theorem, this number is equal to $\frac{|G|}{|C_{a_i}|}$. This is summed up in the following formula :

$$\mathcal{N}_g(C_1, \cdots C_k) = \sum_{A_1, \cdots A_g} \frac{|G|}{|A_1|} \cdots \frac{|G|}{|A_g|} \times \\ \mathcal{N}_0(A_1, A_1^{-1}, \cdots A_g, A_g^{-1}, C_1 \cdots C_k) \quad (1)$$

where the sum is over all conjugacy classes of G.

The only thing left to compute is now $\mathcal{N}_0(A_1, A_1^{-1}, \cdots A_g, A_g^{-1}, C_1 \cdots C_k)$. In order to simplify the notations, we'll just compute $\mathcal{N}_0(C_1, \cdots C_k)$. we'll use Shur's lemma for this. Denote $e_{C_i} = \sum_{g \in C_i} e_g$. The way we obtain the formula is that we compute the trace of $e_{C_1} \cdots e_{C_k}$ acting on $\mathbb{C}[G]$ in two different ways, corresponding to the two sides in the theorem A.8. On one side, we have that for $g \in G$, $g \neq 1$, $\mathrm{Tr}(g) = 0$, and $\mathrm{Tr}(1) = |G|$. Expanding the product in $e_{C_1} \cdots e_{C_k}$ gives us

$$\mathrm{Tr}(e_{C_1} \cdots e_{C_k}) = |G| \mathcal{N}_0(C_1, \cdots C_k)$$

.

Let now V be a irreducible representation of G. We can consider e_C as a morphism $V \to V$ as a sum of linear transforms. Because it is central, e_C is actually a morphism of representations, and hence, by A.9 it acts on V by multiplication by a scalar λ. We can compute λ by computing its trace in the following manner :

$$\lambda \dim V = \sum_{g \in C} \mathrm{Tr}(g) = |C| \chi(C)$$

and hence

$$\lambda = \frac{|C| \chi(C)}{\chi(1)}.$$

So, on the right hand side of A.8, we have

$$\mathrm{Tr}(e_{C_1} \cdots e_{C_k}) = \sum_{\chi \text{ irreducible}} \chi(1)^2 \prod_{i=1}^{k} \frac{|C| \chi(C)}{\chi(1)}$$

Putting everything back into 1, we get :

$$\begin{aligned}
\mathcal{N}_g(C_1, \cdots C_k) = \sum_{A_1, \cdots A_g} \frac{|G|}{|A_1|} \cdots \frac{|G|}{|A_g|} \times \frac{|A_1||A_1^{-1}| \cdots |A_g||A_g^{-1}||C_1| \cdots |C_k|}{|G|} \times \\
\sum_{\chi \text{ irreducible}} \frac{\chi(A_1)\chi(A_1^{-1}) \cdots \chi(A_g)\chi(A_g^{-1})\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}} \\
= |G|^{g-1} |C_1| \cdots |C_k| \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}} \left(\sum_{i=1}^{g} |A_i| \chi(A_i) \overline{\chi(A_i)} \right) \\
= |G|^{2g-1} |C_1| \cdots |C_k| \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}}
\end{aligned} \tag{2}$$

where the last line is due to the orthogonality of characters, and gives the desired result. □

B Code

Code B.1 (Code for $|\chi_\lambda(p,\cdots,p)|$).

```
def chi_lambda(lbda,p): #lbda partition of n
    n=sum(lbda)
    nb_p = n//p
    L=[p for k in range(nb_p)]
    coeur = Partition(lbda).core(p)
    s=0
    if coeur ==[]:
        s=1
        quotient = Partition(lbda).quotient(p)
        for lbd in quotient:
            s*= dimension(lbd)
            s/= factorial(sum(lbd))
        s*= factorial(n/p)
    return s
```

Code B.2 (Code for the sign of $\chi_\lambda(p,\cdots,p)$).

```
def sign_chi_lambda(lbda,p):
    if lbda==[]:
        return 1
    n=sum(lbda)
    r=len(lbda)
    list_pi = [lbda[k]-(k+1) for k in range(r)]
     #rajouter toutes les boules noires ensuite

    rajouter = lbda[-1]
    dernier = list_pi[-1]
    for k in range(p+1):
        list_pi.append(dernier-rajouter-(k+1))
    sum_heights = 0
    while True:
        check = True
        copy = list_pi[:]
        for k in list_pi:
            if k-p not in list_pi and k>dernier-rajouter-1:
                height = len([x for x in copy if k-p<x<k])
                copy.remove(k)
                copy.append(k-p)
                sum_heights+=height
                check=False

        copy.sort(reverse=True)
        list_pi = copy[:]
```

```
        if check:
            break
    return (-1)^sum_heights
```

Code B.3 (Murnaghan-Nakayama for μ).

```
def dimension(lbda): #gives dimension of representation associated with lbda
    l_lambda = len(lbda)
    n=sum(lbda)
    res = factorial(n)
    for i in range(l_lambda):
        for j in range(i+1,l_lambda):
            res*= (lbda[i]-lbda[j]-(i+1)+(j+1))
    for i in range(l_lambda):
        res/= factorial(lbda[i]+l_lambda-(i+1))
    return res


def from_pi_to_part(L): #L sorted
    res=[]
    for k in range(len(L)):
        rajouter = L[k]+k+1
        if rajouter >0:
            res.append(rajouter)
    return res


def remove_all_strips(mui,lbda): #mui is an integer, lambda a partition of n
    n = sum(lbda)
    r = len(lbda)
    list_pi = [lbda[k]-(k+1) for k in range(r)]

    #adding black beads

    rajouter = lbda[-1]
    dernier = list_pi[-1]

    for k in range(mui+1):
        list_pi.append(dernier-rajouter-(k+1))
    resultat=[] #first is the new partition, secondly the height

    for k in list_pi:
        if k-mui not in list_pi and k>=dernier:
            new_list = list_pi[:]
            height = len([p for p in list_pi if k-mui<p<k])
            new_list.remove(k)
            new_list.append(k-mui)
```

```
            new_list.sort(reverse=True)
            new_part = from_pi_to_part(new_list)
            resultat.append([new_part, height])

    return resultat

def murnaghan_nakayama(lbda,mu,N): #lbda partition, new_mu of the form [mu,N,N,N
    if mu==[]:
        return sign_chi_lambda(lbda,N)*chi_lambda(lbda,N)
    valeur_finale =0
    mui = mu[0]
    enlever = remove_all_strips(mui,lbda)
    new_mu=mu[:]
    new_mu.remove(mui)
    for (new_part,height) in enlever:
        valeur_finale+=(-1)^height*murnaghan_nakayama(new_part,new_mu,N)
    return valeur_finale
```

Code B.4.

```
def HN(N,mu,nb_termes):  #I def HN( empty | empty empty mu) , return the result
    serie = [0 for k in range(nb_termes)]
    d = sum(mu) # N divides d
    for p in range(d//N,nb_termes):
        k=N*p
        Sk = SymmetricGroup(k)


        liste_part = list(Partitions(k))


        eta_cycle_type = [N for l in range(p)]
        index_eta = liste_part.index(eta_cycle_type)

        eta_mu_cycle_type =mu[:]
        for osef in range(p-(d//N)):
            eta_mu_cycle_type.append(N)
        eta_mu_cycle_type.sort(reverse=True)
        index_eta_mu = liste_part.index(eta_mu_cycle_type)

        C1 = Sk.conjugacy_class(eta_cycle_type)
        C2 = Sk.conjugacy_class(eta_mu_cycle_type)

        L1 = [murnaghan_nakayama(lbd,[],N) for lbd in liste_part]
        L2 = [murnaghan_nakayama(lbd,mu,N) for lbd in liste_part]
        L_dim = [dimension(lbd) for lbd in liste_part]
        nb_part = len(L1)
```

```
        s=0
        for i in range(nb_part):
            s+=L1[i]^2*L2[i]/L_dim[i]

        serie[p]= s
        serie[p]*=(C1.cardinality())^2*(C2.cardinality())/factorial(k)^2
    return serie

def HN_point(N,mu,nb_termes):
    serie_vide = HN(N,[],nb_termes)
    serie_mu = HN(N,mu,nb_termes)
    R.<X> = PowerSeriesRing(QQ)
    serie_vide_R = R(serie_vide)
    serie_mu_R = R(serie_mu)
    res = list(serie_vide_R^(-1)*(serie_mu_R))[:nb_termes]

    return res
```

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